Instantons on Cylindrical Manifolds

Teng Huang

Abstract

We consider an instanton, $\mathbf{A}$, with L^2-curvature $F_{\mathbf{A}}$ on the cylindrical manifold $Z = \mathbf{R} \times M$, where M is a closed Riemannian n-manifold, $n \geq 4$. We assume M admits a 3-form P and a 4-form Q satisfy $dP = 4Q$ and $d *_M Q = (n-3) *_M P$. Manifolds with these forms include nearly Kähler 6-manifolds and nearly parallel G_2-manifolds in dimension 7. Then we can prove that the instanton must be a flat connection.

Keywords. instantons, special holonomy manifolds, Yang-Mills connection

1 Introduction

Let X be an $(n+1)$-dimensional Riemannian manifold, G be a compact Lie group and E be a principal G-bundle on X. Let A denote a connection on E with the curvature F_A. The instanton equation on X can be introduced as follows. Assume there is a 4-form Q on X. Then an $(n-3)$-form $*Q$ exists, where $*$ is the Hodge operator on X. A connection, A , is called an anti-self-dual instanton, when it satisfies the instanton equation

$$* F_A + *Q \wedge F_A = 0 \tag{1.1}$$

When $n+1 > 4$, these equations can be defined on the manifold X with a special holonomy group, i.e. the holonomy group G of the Levi-Civita connection on the tangent bundle TX is a subgroup of the group $SO(n+1)$. Each solution of equation(1.1) satisfies the Yang-Mills equation. The instanton equation (1.1) is also well-defined on a manifold X with non-integrable G-structures, but equation (1.1) implies the Yang-Mills equation will have torsion.

T. Huang (corresponding author): Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, PR China; e-mail: oula143@mail.ustc.edu.cn

Mathematics Subject Classification (2010): 53C07,58E15

Instantons on the higher dimension, proposed in [6] and studied in [5, 8, 9, 22, 23], are important both in mathematics [9, 22] and string theory [12]. In this paper, we consider the cylinder manifold $Z = \mathbf{R} \times M$ with metric

$$g_Z = dt^2 + g_M$$

where M is a compact Riemannian manifold. We assume M admits a 3-form P and a 4-form Q satisfying

$$dP = 4Q \tag{1.2}$$

$$d *_M Q = (n-3) *_M P. \tag{1.3}$$

On Z,the 4-form [14, 15] can be defined as

$$Q_Z = dt \wedge P + Q.$$

Then the instanton equation on the cylinder manifold Z is

$$* F_{\mathbf{A}} + *Q_Z \wedge F_{\mathbf{A}} = 0 \tag{1.4}$$

Remark 1.1. Manifolds with P and Q satisfying equations (1.2), (1.3) include nearly Kähler 6-manifolds and nearly parallel G_2-manifolds.

(1)M is a nearly Kähler 6-manifold. It is defined as a manifold with a 2-form ω and a 3-form P such that

$$d\omega = 3 *_M P \; and \; dP = 2\omega \wedge \omega =: 4Q$$

For a local orthonormal co-frame $\{e^a\}$ on M one can choose

$$\omega = e^{12} + e^{34} + e^{56} \; and \; P = e^{135} + e^{164} - e^{236} - e^{245},$$

where $a = 1, \dots, 6$, $e^{a_1 \dots a_l} = e^1 \wedge \dots e^l$, and

$$*_M P = e^{145} + e^{235} + e^{136} - e^{246}, \; Q = e^{1234} + e^{1256} + e^{3456}.$$

Here $*_M$ denotes the $*$-operator on M.

(2)M is a nearly parallel G_2 manifold. It is defined as a manifold with a 3-form P (a G_2 structure [4]) preserved by the $G_2 \subset SO(7)$ such that

$$dP = \gamma *_M P$$

for some constant $\gamma \in \mathbf{R}$. For a local orthonormal co-frame e^a, $a = 1, \dots, 7$, on M one can choose

$$P = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}$$

and therefore

$$*_M P =: Q = e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} + e^{1256} - e^{1247}.$$

It is easy to check $dP = 4Q$.

Constructions of solutions of the instanton equations on cylinders over nearly Kähler 6-manifolds and nearly parallel G_2 manifold were considered in [1, 13, 15, 16]. In [16] section 4, the authors confirm that the standard Yang-Mills functional is infinite on their solutions. In this paper, we assume the instanton **A** has L^2-bounded curvature $F_{\mathbf{A}}$. Then we have the following theorem.

Theorem 1.2. *(Main theorem) Let $Z = \mathbf{R} \times M$, here M is a closed Riemannian n-manifold, $n \geq 4$, which admits a smooth 3-form P and a smooth 4-form Q satisfying equations (1.2) and (1.3). Let* ***A*** *be a instanton over Z. Assume that the curvature $F_{\mathbf{A}} \in L^2(Z)$ i.e.*

$$\int_Z \langle F_{\mathbf{A}} \wedge *F_{\mathbf{A}} \rangle < +\infty$$

Then the instanton is a flat connection.

2 Esitimation of Curvature of Yang-Mills connection with torsion

Let Q be a smooth 4-form on n-dimensional manifold X. Let A be an anti-self-dual instanton which satisfies the instanton equation (1.1). Taking the exterior derivative of (1.1) and using the Bianchi identity, we obtain

$$d_A * F_A + *\mathcal{H} \wedge F_A = 0, \tag{2.1}$$

where the 3-form $\mathcal{H}$ is defined by

$$*\,\mathcal{H} = d(*Q). \tag{2.2}$$

The second-order equation (2.1) differs from the standard Yang-Mills equation by the last term involving a 3-from $\mathcal{H}$. This torsion term naturally appears in string-theory compactifications with fluxes [2, 10, 11]. For the case $d(*Q) = 0$, the torsion term vanishes and the instanton equation (1.1) imply the Yang-Mills equation. The latter also holds true when the instanton solution A satisfies $d(*Q) \wedge F_A = 0$ as well, like the cases, on nearly Kähler 6-manifolds, nearly parallel G_2-manifolds and Sasakian manifolds [14].

In section 4.2 of [1] or in section 2.1 of [13], they online that in the instanton does not extremize the standard Yang-Mills functional in the torsionful case. Instead, they add a add a Chern-Simons-type term to get the following functional:

$$S(A) = -\int_X Tr\big(F_A \wedge *F_A + F_A \wedge F_A \wedge *Q\big), \tag{2.3}$$

This is the right functional which produces the correct Yang-Mills equation with torsion. And the instanton equations (1.1) can be derived from this action using a Bogomolny

argument. In the case of a closed form $*Q$, the second term in (2.3) is topological invariant and the torsion (2.2) disappears from (2.1).

In this section, we will derive monotonicity formula for Yang-Mills connection with torsion (2.1). Its proof follows Tian's arguments about pure Yang-Mills connection in [22] with some modifications.

Let X be a compact Riemannian n-manifold with metric g and E is a vector bundle over X with compact structure group G. For any connection A of E, its curvature form F_A takes value in $Lie(G)$. The norm of F_A at any $p \in X$ is given by

$$|F_A|^2 = \sum_{i,j=1}^{n} \langle F_A(e_i, e_j), F_A(e_i, e_j)\rangle,$$

where $\{e_i\}$ is any orthonormal basis of T_pX, and $\langle \cdot, \cdot\rangle$ is the Killing form of $Lie(G)$.

As in [22], we consider a one-parameter family of diffeomorphisms $\{\psi_t\}_{|t|<\infty}$ of X with $\psi_0 = id_X$. We fix a connection A_0, and denote by its derivative D. Then for any connection A, we can define a one-parameter family $\{A_t\}$ in the following way. Let τ_t^0 be the parallel transport on E associated to A_0 along the path $\psi_s(x)_{0<s<t}$, where $x \in X$. More precisely, for any $u \in E_x$ over $x \in X$, let $\tau_s^0(u)$ be the section of E over the path $\psi_s(x)_{0<s<t}$ such that

$$D_{\frac{\partial}{\partial s}} \tau_s^0(u) = 0,\ \tau_0^0(u) = u.$$

We define a family of connections $A_t := \psi_t^*(A)$ by defining its covariant derivative as

$$D_\nu^t s = (\tau_t^0)^{-1}\big(D_{d\psi_t(\nu)}(\tau_t^0(s))\big)$$

for any $\nu \in TX,\ s \in \Gamma(X, E)$. Then the curvature of A_t is written as

$$F_{A_t}(X_1, X_2) = (\tau_t^0)^{-1} \cdot F_A(d\psi_t(X_1), d\psi_t(X_2)) \cdot \tau_t^0.$$

It follows that

$$\int_X |F_A|^2 = \int_X \sum_{i,j=1}^{n} |F_A(d\psi_t(e_i), d\psi_t(e_j))|^2(\psi_t(x))dV_g.$$

where dV_g denotes the volume form of g, and $\{e_i\}$ is any local orthonormal basis of TX. By changing variables, we obtain

$$\int_X |F_A|^2 = \int_X \sum_{i,j=1}^{n} |F_A(d\psi_t(e_i(\psi_t^{-1}(x))), d\psi_t(e_j(\psi_t^{-1}(x)))|^2 Jac(\psi_t^{-1})dV_g.$$

Let ν be the vector field $\frac{\partial \psi_t}{\partial t}|_{t=0}$ on X. Then we deduce from the above that

$$\begin{aligned}\frac{d}{dt}YM(A_t)|_{t=0} &= \int_X \langle i_\nu F_A, d_A^* F_A\rangle \\ &= \int_X Tr\big(i_\nu F_A \wedge F_A \wedge (d * Q)\big) \\ &= \int_X \big(|F_A|^2 div\nu + 4\sum_{i,j=1}^n \langle F_A([\nu, e_i], e_j), F_A(e_i, e_j)\rangle\big)dV_g \\ &= \int_X \big(|F_A|^2 div\nu - 4\sum_{i,j=1}^n \langle F_A(\nabla_{e_i}\nu, e_j), F_A(e_i, e_j)\rangle\big)dV_g\end{aligned} \tag{2.4}$$

Fix any $p \in X$, let r_p be a positive number with following properties: there are normal coordinates $x_1, \ldots, x_n$ in the geodesic ball $B_{r_p}(p)$ of (X, g), such that $p = (0, \ldots, 0)$ and for some constant $c(p)$,

$$|g_{ij} - \delta_{ij}| \le c(p)(|x_1|^2 + \ldots + |x_n|^2),$$

$$|dg_{ij}| \le c(p)\sqrt{|x_1|^2 + \ldots + |x_n|^2},$$

where

$$g_{ij} = g\big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big).$$

Let $r(x) := \sqrt{x_1^2 + \ldots + x_n^2}$ be the distance function from p. Define $\nu(x) = \xi(r)r\frac{\partial}{\partial r}$, where ξ is some smooth function with compact support in $B_{r_p}(p)$. Let $\{e_1, \ldots, e_n\}$ be any orthonormal basis near p such that $e_1 = \frac{\partial}{\partial r}$. Since $x_1, \ldots, x_n$ are normal coordinates, we have $\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0$. It follows that

$$\nabla_{\frac{\partial}{\partial r}}\nu = (\xi r)'\frac{\partial}{\partial r} = (\xi' r + \xi)\frac{\partial}{\partial r}. \tag{2.5}$$

Moreover, for $i \ge 2$,

$$\nabla_{e_i}\nu = \xi r(\nabla_{e_i}\frac{\partial}{\partial r}) = \xi\sum_{j=1}^n b_{ij}e_j. \tag{2.6}$$

where $|b_{ij} - \delta_{ij}| = O(1)c(p)r^2$. Applying (2.5) and (2.6) to the variation formula (2.4), we obtain

$$\begin{aligned}\int_X \langle i_\nu F_A, d_A^* F_A\rangle = &\int_X \big(|F_A|^2(\xi' r + (n-4)\xi + O(1)c(p)r^2\xi)dV_g \\ &- 4\int_X (\xi' r|\frac{\partial}{\partial r}\lrcorner F_A|^2)dV_g.\end{aligned} \tag{2.7}$$

where $\frac{\partial}{\partial r}\lrcorner F_A = F_A(\frac{\partial}{\partial r}, \cdot)$.

We choose, for any τ small enough. $\xi(r)=\xi_\tau(r)=\eta(\frac{r}{\tau})$, where η is smooth and satisfies: $\eta(r)=1$ for $r\in[0,1]$, $\eta(r)=0$ for $r\in[1+\epsilon,\infty)$, $\epsilon<0$ and $\eta'(r)\le 0$. Then

$$\tau\frac{\partial}{\partial\tau}(\xi_\tau(r))=-r\xi'_\tau(r). \tag{2.8}$$

Plugging this into (2.7), we obtain

$$\begin{aligned}\int_X\langle i_\nu F_A,d_A^*F_A\rangle&=\tau\frac{\partial}{\partial\tau}\big(\int_X\xi_\tau|F_A|^2dV_g\big)\\&+\big((4-n)+O(1)c(p)\tau^2\big)\int_X\xi_\tau|F_A|^2dV_g\\&-4\tau\frac{\partial}{\partial\tau}\big(\int_X\xi_\tau|\frac{\partial}{\partial r}\lrcorner F_A|^2dV_g\big)\end{aligned} \tag{2.9}$$

Choose a nonngeative number $a\ge O(1)c(p)r_p+\max_{x\in X}|d(*Q)|(x)$. Then we deduce from the above

$$\begin{aligned}&\frac{\partial}{\partial\tau}\big(\tau^{4-n}e^{a\tau}\int_X\xi_\tau|F_A|^2dV_g)\\&=\tau^{4-n}e^{a\tau}\left(4\frac{\partial}{\partial\tau}\big(\int_X\xi_\tau|\frac{\partial}{\partial r}\lrcorner F_A|^2dV_g\big)+(-O(1)c(p)\tau+a)\int_X\xi_\tau|F_A|^2dV_g\right)\\&-e^{a\tau}\tau^{3-n}\int_X Tr(i_\nu F_A\wedge F_A\wedge d(*Q))\end{aligned} \tag{2.10}$$

We have the fact:

$$\begin{aligned}|\int_X Tr(i_\nu F_A\wedge F_A\wedge d(*Q))|&\le\int_X|\nu|\cdot|F_A|^2\cdot\max_{x\in X}|d(*Q)|\\&\le\max_{x\in X}|d(*Q)|\int_{B_{\tau(1+\epsilon)}(p)}\tau(1+\epsilon)|F_A|^2dV_g\end{aligned} \tag{2.11}$$

Then, by integrating on τ and letting ϵ tend to zero, we have already proved:

Theorem 2.1. *Let $r_p,\ c(p)$ and a be as above. Then for any $0<\sigma<\rho<r_p$, we have*

$$\begin{aligned}&\rho^{4-n}e^{a\rho}\int_{B_\rho(p)}|F_A|^2dV_g-\sigma^{4-n}e^{a\sigma}\int_{B_\sigma(p)}|F_A|^2dV_g\\&\ge 4\int_{B_\rho(p)\backslash B_\sigma(p)}r^{4-n}e^{ar}|\frac{\partial}{\partial r}\lrcorner F_A|^2dV_g\\&+\int_\sigma^\rho\big(e^{a\tau}\tau^{4-n}(a-\max_{x\in X}|d(*Q)|-O(1)c_p\tau)\int_{B_\tau(p)}|F_A|^2dV_g\big)d\tau\end{aligned} \tag{2.12}$$

Next, we prove a mean value inequality about the energy dense $|F_A|^2$. The Bochner-Weitzenböck formula ([3], Theorem 3.1) is

$$(d_Ad_A^*+d_A^*d_A)F_A=\nabla_A^*\nabla_AF_A+F_A\circ(Ric\wedge g+2R)+\mathcal{R}^A(F_A).$$

Since A is a instanton and the Bianchi identity $d_AF_A = 0$, we re-write the left hand

$$(d_Ad_A^* + d_A^*d_A)F_A = d_A\big(* (d(*Q) \wedge F_A)\big)$$

Hence

$$|d_A\big(* (d(*Q) \wedge F_A)\big)| \leq n|\nabla(d(*Q))|F_A| + n|d(*Q)||\nabla_AF_A|$$

due to for all $X_1, X_2, \ldots, X_{n-2} \in T_xX$, where $n = dimX$,

$$\begin{aligned} & d_A^*\big((d(*Q) \wedge F_A)\big)(X_1, X_2, \ldots, X_{n-2}) \\ =& -\sum_i (\nabla_A)_{e_i}(d(*Q) \wedge F_A)(e_i, X_1, \ldots, X_{n-2}), \\ =& -\sum_i \big(\nabla_{e_i}(d(*Q)) \wedge F_A\big)(e_i, X_1, \ldots, X_{n-2}) \\ & -\sum_i \big((d(*Q)) \wedge (\nabla_A)_{e_i}F_A\big)(e_i, X_1, \ldots, X_{n-2}), \end{aligned}$$

Then we have

$$\begin{aligned} |\langle (d_Ad_A^* + d_A^*d_A)F_A, F_A\rangle| &\leq n|\nabla(d(*Q))| \cdot |F_A|^2 + n|d(*Q)| \cdot |\nabla_AF_A| \cdot |F_A| \\ &\leq C_1|F_A|^2 + C_2(\varepsilon|\nabla_AF_A|^2 + \frac{1}{\varepsilon}|F_A|^2) \end{aligned} \tag{2.13}$$

where ε is a positive constant.

The quadratic $\mathcal{R}^A(F_A) \in \Omega^2(\mathfrak{g})$ can be expressed with the help of a local orthonormal frame $(e_1, e_2, \ldots, e_n)$ of TX as

$$\mathcal{R}^A(F_A)(X_1, X_2) = 2\sum_{j=1}^{n}[F_A(e_j, X_1), F_A(e_j, X_2)].$$

The estimate of the Laplacian now follow from

$$\begin{aligned} -\nabla^*\nabla|F_A|^2 &= -2|\nabla_AF_A|^2 - 2\langle F_A, \nabla_A^*\nabla_AF_A\rangle \\ &\leq 2\langle F_A, F_A \circ (Ric \wedge g + 2R)\rangle + 2\langle F_A, \mathcal{R}^A(F_A)\rangle \\ &+ C_1|F_A|^2 + C_2(\varepsilon|\nabla_AF_A|^2 + \frac{1}{\varepsilon}|F_A|^2) - 2|\nabla_AF_A|^2 \end{aligned}$$

We choose ε small enough such that $C_2\varepsilon < 2$, then we have

$$\Delta|F_A|^2 \leq C|F_A|^2 + c|F_A|^3.$$

Thus, we get

$$\Delta|F_A| \leq C|F_A| + c|F_A|^2. \tag{2.14}$$

Theorem 2.2. *Let A be any Yang-Mills connection with torsion of a G-bundle E over X. Then there exist constants $\varepsilon = \varepsilon(X, n, Q) > 0$ and $C = C(X, n)$, such that for any $p \in X$ and $p < r_p$, whenever*

$$\rho^{4-n} \int_{B_\rho(p)} |F_A|^2 dV_g \leq \varepsilon$$

then

$$|F_A|(p) \leq \frac{C}{\rho^2} \big(\rho^{4-n} \int_{B_\rho(p)} |F_A|^2 dV_g\big)^{\frac{1}{2}}. \tag{2.15}$$

Our proof here use G.Tian's arguments in [22] for pure Yang-Mills connection.

Proof. By scalling, we may assume that $\rho = 1$. Define a function

$$f(r) = (1-2r)^2 \sup_{x \in B_r(p)} |F_A|(x),\ r \in [0, \frac{1}{2}].$$

Then $f(r)$ is continuous in $[0, \frac{1}{2}]$ with $f(\frac{1}{2}) = 0$, and f attains its maximum at a certain r_0 in $[0, \frac{1}{2}]$.

First we claim that $f(r_0) \leq 64$ if ε is sufficiently small. Assume that $f(r_0) > 64$. Put $b = \sup_{x \in B_{r_0}(p)} |F_A|(x) = |F_A|(x_0)$ by taking $\sigma = \frac{1}{4}(1 - 2r_0)$, we get

$$\begin{aligned} \sup_{x \in B_\sigma(x_0)} |F_A| &\leq \sup_{x \in B_{r_0+\sigma}(p)} |F_A|(x) \\ &\leq \frac{(1-2r_0)^2}{(1-2r_0-2\sigma)^2} \sup_{x \in B_{r_0}(p)} |F_A|(x) = 4b. \end{aligned} \tag{2.16}$$

Clearly, $16\sigma^2 b \geq 64$; i.e., $\sigma\sqrt{b} \geq 2$. Define a scaled metric $\widetilde{g} = bg$. Then the norm $|F_A|_{\widetilde{g}}$ of F_A is equal to $b^{-1}F_A$ with respect to $\widetilde{g}$. Hence

$$\sup_{x \in B_2(x_0, \widetilde{g})} |F_A|_{\widetilde{g}} \leq 4, \tag{2.17}$$

where $B_2(x_0, \widetilde{g})$ denotes the geodesic ball of $\widetilde{g}$ with radius 2 and centered at x_0. Using (2.17), we deduce from (2.14) that in $B_2(x_0, \widetilde{g})$,

$$\Delta_{\widetilde{g}} |F_A|_{\widetilde{g}} \leq (C + 4c)|F_A|_{\widetilde{g}}. \tag{2.18}$$

Then, by using the mean-value theorem, we obtain

$$1 = |F_A|_{\widetilde{g}}(x_0) \leq \widetilde{c}\big(\int_{B_1(x, \widetilde{g})} |F_A|^2_{\widetilde{g}} dV_{\widetilde{g}}\big)^{\frac{1}{2}}. \tag{2.19}$$

where $\widetilde{c}$ is some uniform constant.

However, by the monotonicity (Theorem 2.1),

$$\begin{aligned}\int_{B_1(x_0,\widetilde{g})}|F_A|^2_{\widetilde{g}}dV_{\widetilde{g}} &= (\sqrt{b})^{n-4}\int_{B_{\frac{1}{\sqrt{b}}}(x_0)}|F_A|^2dV_g\\ &\leq (\frac{1}{2})^{4-n}e^{\frac{a}{2}}\int_{B_{\frac{1}{2}}(x_0)}|F_A|^2dV_g\\ &\leq \varepsilon 2^{n-4}e^{\frac{a}{2}}\end{aligned}$$

Combining this with (2.19), we obtain

$$1 \leq \widetilde{c}\varepsilon 2^{n-4}e^{\frac{a}{2}}.$$

It is impossible since we can choose $\varepsilon = \varepsilon(X, n, Q)$ sufficiently small. The claim is proved.

Thus, we have

$$\sup_{x\in B_{\frac{1}{4}}(p)}|F_A|(x) \leq 4f(r_0) \leq 256.$$

It follows from this and (2.14) with $\widetilde{g}$ replaced by g that for some uniform constant c',

$$\Delta_g|F_A| \leq c'|F_A|. \tag{2.20}$$

Then (2.15) follows from (2.20) and a standard Moser iteration. □

3 Asymptotic Behavior and Conformal Transformation

3.1 Chern-Simons Functional

The main aim of this section is to get the relationship between gauge theory on an n-dimensional manifold M and the gauge theory on the $n+1$-dimensional manifold $Z = \mathbf{R}\times M$. The main idea is that a connection on $\mathbf{R}\times M$ can be regard as one-parameter families of connections on M by local trivialisation. Let t be the standard parameter on the factor $\mathbf{R}$ in the $\mathbf{R}\times M$ and let $\{x^j\}_{j=1}^n$ be local coordinates of M. A connection $\mathbf{A}$ over the cylinder Z is given by a local connection matrix

$$\mathbf{A} = A_0dt + \sum_{i=1}^n A_idx^i.$$

where A_0 and A_i dependence on all $n+1$ variable $t, x^1, \ldots, x^n$. We take $A_0 = 0$ (sometimes called a temporal gauge). In this situation, the curvature in a mixed x_i-plane is given by the simple formula

$$F_{0i} = \frac{\partial A_i}{\partial t}.$$

We denote $A=\sum_{i=1}^{n}A_i dx^i$ and $\dot{A}=\frac{\partial A}{\partial t}$, then the curvature is given by

$$F_{\mathbf{A}}=F_A+dt\wedge\dot{A}.$$

M has a Riemannian metric and $*$-operator $*_M$. If ϕ is a 1-form on M then, for $*$-operator defined on $Z=\mathbf{R}\times M$ with respect to the product metric, we have

$$*(dt\wedge\phi)=*_M\phi.$$

Then the instanton equation is equivalent to

$$*_M\dot{A}=-*_M P\wedge F_A, \tag{3.1}$$

$$*_M F_A=-\dot{A}\wedge *_M P-*_M Q\wedge F_A. \tag{3.2}$$

Let $E\to M$ be a vector bundle, the space $\mathcal{A}$ is an affine space modelled on $\Omega^1(\mathfrak{g}_E)$ so, fixing a reference connection $A_0\in\mathcal{A}$, we have

$$\mathcal{A}=A_0+\Omega^1(\mathfrak{g}_E).$$

We define the Chern-Simons functional by

$$CS(A):=-\int_M Tr\big(a\wedge d_{A_0}a+\frac{2}{3}a\wedge a\wedge a\big)\wedge *_M P,$$

fixing $CS(A_0)=0$. This functional is obtained by integrating of the Chern-Simons 1-form

$$\Gamma(\beta)_A=\Gamma_A(\beta_A)=-2\int_M Tr(F_A\wedge\beta_A)\wedge *_M P.$$

We find CS explicitly by integrating Γ over paths $A(t)=A_0+ta$, from A_0 to any $A=A_0+a$:

$$\begin{aligned}CS(A)-CS(A_0)&=\int_0^1\Gamma_{A(t)}\big(\dot{A}(t)\big)dt\\&=-2\int_0^1\Big(\int_M Tr\big((F_{A_0}+td_{A_0}a+t^2a\wedge a)\wedge a\big)\wedge *_M P\Big)dt\\&=-\int_M Tr\big(d_{A_0}a\wedge a+\frac{2}{3}a\wedge a\wedge a\big)\wedge *_M P+C,\end{aligned}$$

where $C=C(A_0,a)$ is a constant and vanishes if A_0 is an instanton. The co-closed condition $d*_M P=0$ implies that the Chern-Simons 1-form is closed. So it does not depend on the path $A(t)$ [7, 20, 21]. Since

$$dTr(d_{A_0}a\wedge a+\frac{2}{3}a\wedge a\wedge a)=Tr(F^2_{A_0+a}-F^2_{A_0}),$$

we can re-write Chern-Simons functional as

$$\begin{aligned}CS(A)-CS(A_0)&=-\int_M Tr\big(d_{A_0}a\wedge a+\frac{2}{3}a\wedge a\wedge a\big)\wedge *_M P\\&=-\frac{1}{n-3}\int_M Tr(F_A^2-F^2_{A_0})\wedge *_M Q,\end{aligned} \tag{3.3}$$

the second formula holds because of equation (1.3).

3.2 Asymptotic Behavior

Let $Z = \mathbf{R}\times M$ be an $(n+1)$-manifold and M be an n-manifold. Let **A** be an instanton on Z with finite energy, i.e. $\int_Z |F_{\mathbf{A}}|^2 < \infty$. We use the Chern-Simons functional to study the decay of instantons over the cylinder manifold. We will see that, an instanton with $L^2(Z)$-bounded curvature can be represented by a connection form which decays exponentially on the tube.

We consider a family of bands $B_T = (T-1,T)\times M$ which we identify with the model $B = (0,1)\times M$ by translation. So the integrability of $|F_{\mathbf{A}}|^2$ over the end implies that

$$\int_{(T,T+1)\times M} |F_{\mathbf{A}}|^2 \to 0 \quad as \quad T\to\infty.$$

Proposition 3.1. *Let $Z = \mathbf{R}\times M$, here M is a closed Riemannian n-manifold, $n\geq 4$, which admits a smooth 3-form P and a smooth 4-form Q those satisfy equations (1.2) and (1.3). Let* **A** *be a instanton over Z, then at the end of Z there is a flat connection Γ over M such that* **A** *converges to Γ, i.e. the restriction* $\mathbf{A}|_{M\times\{T\}}$ *converges (modulo gauge equivalence) in C^∞ over M as $T\to\infty$.*

Proof. We choose

$$\rho = \frac{1}{2} Inj\big((T,T+1)\times M, g_Z\big),$$

where $Inj\big((t,t+1)\times M\big) > 0$ denotes the injectivity radius of the manifold $\big((T,T+1)\times M, g_Z\big)$. It's easy to see ρ is not dependent on t. Since $*Q_Z = *_M P + dt\wedge *_M Q$, we obtain

$$\max_{(x,t)\in Z} |d(*Q_Z)|^2 = \max_{x\in M}\big(|d*_M P|^2 + |d(*_M Q)|^2\big) < \infty,$$

and

$$\max_{(x,t)\in Z} |\nabla(d*Q_Z)| \leq \max_{x\in M}|\nabla(d*_M P)| + \max_{x\in M}|\nabla(d*_M Q)| < \infty.$$

We denote $\varepsilon = \varepsilon(Z,n,Q)$ as the constant in Theorem 2.2. Then there exist T sufficiently large such that $t\geq T$, we have

$$\int_{(T,T+1)\times M} |F_{\mathbf{A}}|^2 \leq \varepsilon\rho^{n-3},$$

Then for any point $(t,x)\in(T,T+1)\times M$, we have

$$\rho^{3-n}\int_{B_\rho(x,t)} |F_{\mathbf{A}}|^2 \leq \varepsilon.$$

From Theorem 2.2, we have

$$|F_{\mathbf{A}}|(t,x) \leq \frac{C}{\rho^2}\big(\rho^{3-n}\int_{B_\rho(x,t)}|F_{\mathbf{A}}|^2\big)^{\frac{1}{2}}$$

It implies that for any sequence $T_i \to \infty$ there exist a flat connection Γ over M such that, after suitable gauge transformations,

$$\mathbf{A}_{T_i} \to \Gamma,$$

in C^∞ over M. ☐

Under above, from (3.3) we can write Chern-Simons function as

$$CS(A(T)) - CS(A(\infty)) = -\frac{1}{n-3}\int_M Tr(F_A \wedge F_A) \wedge *_M Q.$$

Lemma 3.2. *Let* **A** *be an instanton with temporal gauge, then*

$$\begin{aligned} CS(A(T')) - CS(A(T)) = &\int_{[T,T']\times M} Tr(F_{\mathbf{A}} \wedge *F_{\mathbf{A}}) \\ &- (n-3)\int_T^{T'} \big(CS(A(t)) - CS(A_\infty)\big)dt \end{aligned} \tag{3.4}$$

Proof. Using the method of previous section, we have

$$\frac{d}{dt}CS\big(A(t)\big) = \Gamma_{A(t)}(\dot{A(t)})$$

Then

$$\begin{aligned} CS(A(T')) - CS(A(T)) &= \int_T^{T'} dCS\big(A(t)\big) = \int_T^{T'} \Gamma_{A(t)}\big(\dot{A}(t)\big)dt \\ = -2\int_{[T,T']\times M} Tr(F_{A(t)} \wedge dt \wedge \dot{A}(t)) \wedge *_M P \\ = -\int_{[T,T']\times M} Tr(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \wedge *Q_Z + \int_T^{T'}\Big(\int_M Tr(F_A \wedge F_A) \wedge *_M Q\Big)dt \\ = \int_{[T,T']\times M} Tr(F_{\mathbf{A}} \wedge *F_{\mathbf{A}}) - (n-3)\int_T^{T'}\big(CS(A(t)) - CS(A(\infty))\big)dt \end{aligned}$$

☐

We set

$$J(T) = \int_T^\infty \|F_{\mathbf{A}}\|^2_{L^2} = -\int_{[T,\infty)\times M} Tr(F_{\mathbf{A}} \wedge *F_{\mathbf{A}}).$$

On the one hand, we can express $J(T)$ as the integration of $Tr(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \wedge *Q_Z$, since **A** is an instanton.

$$J(T) = \int_{[T,\infty)\times M} Tr(F_{\mathbf{A}} \wedge F_{\mathbf{A}}) \wedge *Q_Z$$

From (3.4), taking the limit over finite tubes $(T, T') \times M$ with $T' \to +\infty$ we see that

$$J(T) = CS(A(T)) - CS(A_\infty) - (n-3)\int_T^\infty \big(CS(A(t)) - CS(A(\infty))\big)dt \tag{3.5}$$

where $A(T)$ is the connection over M obtain by restriction to $M \times \{T\}$. From (3.5), we can obtain the T derivative of J as

$$\frac{d}{dT}J(T) = \frac{d}{dT}\big(CS(A(T)) - CS(A(\infty))\big) + (n-3)\big(CS(A(T)) - CS(A_\infty)\big) \quad (3.6)$$

On the other hand, the T derivative of $J(T)$ can be expressed as minus the integration over $M \times \{T\}$ of the curvature density $|F_{\mathbf{A}}|^2$, and this is exactly the n-dimensional curvature density $|F_{A(T)}|^2$ plusing the density $|\dot{A}|^2$. By the relation (1.2) and (1.3) between the two components of the curvature for an instanton, we have

$$\begin{aligned}
\|F_{A(T)}\|^2_{L^2(M)} &= -\int_M Tr(F_{A(T)} \wedge *_M F_{A(T)}) \\
&= -\int_M Tr\big(F_{A(T)} \wedge (-\dot{A}(T) \wedge *_M P)\big) \\
&\quad + \int_M Tr(F_{A(T)} \wedge F_{A(T)}) \wedge *_M Q \\
&= \|\dot{A}\|^2_{L^2(M)} - (n-3)\big(CS(A(T)) - CS(A_\infty)\big)
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dT}J(T) &= -2\|F_{A(T)}\|^2_{L^2(M)} - (n-3)\big(CS(A(T)) - CS(A_\infty)\big) && (3.7)\\
&= -2\|\dot{A}\|^2_{L^2(M)} + (n-3)\big(CS(A(t)) - CS(A_\infty)\big) && (3.8)
\end{aligned}$$

From (3.6) and (3.7), we have

$$\frac{d}{dT}\big(CS(A(T)) - CS(A_\infty)\big) + 2(n-3)\big(CS(A(T)) - CS(A_\infty)\big) \leq 0$$

From (3.6) and (3.8), we have

$$\frac{d}{dT}\big(CS(A(T)) - CS(A_\infty)\big) \leq 0$$

It's easy to see these imply that $\big(CS(A(t)) - CS(A_\infty)\big)$ is non-negative and decays exponentially,

$$0 \leq \big(CS(A(T)) - CS(A_\infty)\big) \leq Ce^{-(2n-6)T} \quad (3.9)$$

We introduce a parameter δ and set

$$L_\delta(T) := \int_T^\infty e^{\delta t}\|F_{\mathbf{A}}\|^2_{L^2(M)}dt$$

Theorem 3.3. *Let **A** be an instanton with L^2-bounded curvature on $Z = \mathbf{R} \times M$, here M is a closed Riemannian n-manifold, $n \geq 4$, which admits a smooth 3-form P and a smooth 4-form Q those satisfy equations (1.2) and (1.3). Then there is a constant C, such that*

$$L_\delta(t) \leq Ce^{(\delta-2n+6)t}$$

where $0 < \delta < 2n-6$.

Proof. From (3.6), we get

$$\|F_{\mathbf{A}}\|^2_{L^2(M)} = -\frac{d}{dt}\big(CS(A(t)) - CS(A_\infty)\big) - (n-3)\big(CS(A(t)) - CS(A_\infty)\big)$$

Then

$$\begin{aligned} L_\delta(T) &= -\int_T^\infty e^{\delta t}\frac{d}{dt}\big(CS(A(t)) - CS(A_\infty)\big) \\ &\quad - (n-3)\int_T^\infty e^{\delta t}\big(CS(A(t)) - CS(A_\infty)\big) \\ &= -e^{\delta t}\big(CS(A(t)) - CS(A_\infty)\big)|_T^\infty + \int_T^\infty \delta e^{\delta t}\big(CS(A(t)) - CS(A_\infty)\big) \\ &\quad - (n-3)\int_T^\infty e^{\delta t}\big(CS(A(t)) - CS(A_\infty)\big) \\ &\leq e^{\delta T}\big(CS(A(T)) - CS(A_\infty)\big) + \int_T^\infty \delta e^{\delta t}\big(CS(A(t)) - CS(A_\infty)\big) \\ &\leq Ce^{(\delta-2n+6)T} + \int_T^\infty C\delta e^{(\delta-2n+6)t}dt \\ &= (C + \frac{C\delta}{2n-6-\delta})e^{(\delta-2n+6)T} \end{aligned}$$

□

3.3 Conformal Transformation

We consider $\bar{Z} = C(M)$, where $C(M)$ is a cone over M with metric

$$g_{\bar{Z}} = dr^2 + r^2 g_M = e^{2t}(dt^2 + g_M),$$

where $r := e^t$.

It means that the cone $C(M)$ is conformally equivalent to the cylinder

$$Z = \mathbf{R} \times M$$

with the metric

$$g_Z = dt^2 + g_M.$$

Furthermore, we can show that the instanton equation on the cone $\bar{Z} = C(M)$ is related with the instanton equation on the cylinder $Z = \mathbf{R} \times M$,

$$\bar{*}F_{\mathbf{A}} + \bar{*}Q_{\bar{Z}} \wedge F_{\mathbf{A}} = e^{(n-3)t}(*F_{\mathbf{A}} + *Q_Z \wedge F_{\mathbf{A}}) = 0,$$

where $dimC(M) = dimZ = n+1$, $\bar{*}$ is the $*$-operator in $C(M)$.And

$$Q_{\bar{Z}} = e^{4t}(dt \wedge P + Q). \tag{3.10}$$

In the other word, equation on $C(M)$ is equivalent to the equation on $\mathbf{R}\times M$ after rescaling of the metric. So we can only consider the instanton equation

$$\bar{*}F_{\mathbf{A}}+\bar{*}Q_{\bar{Z}}\wedge F_{\mathbf{A}}=0$$

on the cone $C(M)$ over M. Since

$$\bar{*}_Z Q_{\bar{Z}}=e^{(n-3)}*Q_Z=e^{(n-3)t}(*_M P+dt\wedge *_M Q),$$

by direct calculate, $\bar{*}_Z Q_{\bar{Z}}$ is closed. This implies that the instantons also satisfy the pure Yang-Mills equations with respect to the metric $g_{\bar{Z}}$.

Proposition 3.4. *Let **A** be a instanton on $Z=\mathbf{R}\times M$, here M is a closed Riemannian n-manifold, $n\geq 4$, which admits a smooth 3-form P and a smooth 4-form Q those satisfy equations (1.2) and (1.3). Then the connection A is a Yang-Mills connection on $C(M)$.*

After rescaling of the metric,

$$F_{\mathbf{A}}\wedge\bar{*}F_{\mathbf{A}}=e^{(n-3)t}F_{\mathbf{A}}\wedge *F_{\mathbf{A}}.$$

The curvature $F_{\mathbf{A}}$ is L^2-bounded over Z. We shall prove that the curvature $F_{\mathbf{A}}$ is still L^2-bounded over $C(M)$ by the following lemma.

Lemma 3.5. *Let **A** be a instanton on $Z=\mathbf{R}\times M$ with L^2-bounded curvature $F_{\mathbf{A}}$, i.e.*

$$\int_{\mathbf{R}\times M}\langle F_{\mathbf{A}}\wedge *F_{\mathbf{A}}\rangle<+\infty,$$

here M is a closed Riemannian n-manifold, $n\geq 4$, which admits a 3-form P and a 4-form Q those satisfy equations (1.2) and (1.3). Then

$$\int_{\mathbf{R}\times M}\langle F_{\mathbf{A}}\wedge\bar{*}F_{\mathbf{A}}\rangle<+\infty.$$

Proof. From theorem 3.3, we have

$$L_{n-3}(T)=\int_T^\infty e^{(n-3)t}\|F_{\mathbf{A}}\|^2_{L^2(M)}dt\leq Ce^{-(n-3)T}$$

Then for any constant $T\in[0,\infty)$, we have

$$\begin{aligned}\int_{\mathbf{R}\times M}\langle F_{\mathbf{A}}\wedge\bar{*}F_{\mathbf{A}}\rangle&=\int_{\mathbf{R}\times M}e^{(n-3)t}\langle F_{\mathbf{A}}\wedge *F_{\mathbf{A}}\rangle\\&=\int_{(-\infty,T]\times M}+\int_{[T,\infty)\times M}e^{(n-3)t}\langle F_{\mathbf{A}}\wedge *F_{\mathbf{A}}\rangle\\&\leq e^{(n-3)T}\int_{\mathbf{R}\times M}\langle F_{\mathbf{A}}\wedge *F_{\mathbf{A}}\rangle+L_{n-3}(T)<+\infty\end{aligned}$$

□

We only consider instantons with L^2-bounded curvature on the cone of M. In the next section, we will give a vanishing theorem for Yang-Mills connection with finite energy on the cone of M.

4 Vanishing Theorem for Yang-Mills

In this section, notations may be different from the previous sections. We use the conformal technique to give the vanishing theorem for Yang-Mills connection on the cone of M.

Let M be a Riemannian $n+1$-manifold. Suppose $X\in\Gamma(TM)$ is a conformal vector field on (M,g), namely,

$$\mathcal{L}_X g = 2fg$$

where $f\in C^\infty(M)$. Here $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

The vector field X generates a family of local conformal diffeomorphism.

$$F_t = exp(tX):\ M\to M$$

This family of local conformal diffeomorphism can induce a bundle automorphism, $\tilde{F}_t$, of the principal bundle P. Such a lift is readily obtained from a connection on P by setting $\widetilde{F}_t=\exp(t\widetilde{X})$ where $\widetilde{X}$ is the horizontal lift of X on P. If A is the connection form we have $i_{\widetilde{X}}A=0$ since $\widetilde{X}$ is horizontal. Thus the Lie derivative of A is can be expressed in terms of the curvature F_A: $\mathcal{L}_{\widetilde{X}}A=i_{\widetilde{X}}dA+di_{\widetilde{X}}A=i_{\widetilde{X}}(F_A-\frac{1}{2}[A,A])=i_XF_A$. And hence $\widetilde{F}_t^*A=A+ti_XF_A+o(t^2)$. One can see the detailed process in [19].

We will consider the variation of the Yang-Mills functional under the family of diffeomorphism.

$$YM(A,g)=\int_M |F_A|^2 dVol_g$$

where $dVol_g=\sqrt{detg}dx$ is the volume form of M.

$$\lambda := |F_A|^2 dVol_g$$

is an n-form on M. For any $\eta\in C_0^\infty(M)$

$$\begin{aligned}0&=\int_M d[(i_X\lambda)\eta]=\int_M \eta d(i_X\lambda)+\int_M d\eta\wedge i_X\lambda\\&=\int_M \eta\mathcal{L}_X\lambda+\int_M d\eta\wedge i_X\lambda\end{aligned}$$

that is

$$\int_M \eta\mathcal{L}_X\lambda=-\int_M d\eta\wedge i_X\lambda \tag{4.1}$$

where i_X stands for the inner product with the vector X. Now, let us compute $\mathcal{L}_X\lambda$.

Lemma 4.1. *Let $\lambda=Tr(F_A\wedge *F_A)$ and X be a smooth vector field on M satisfying $\mathcal{L}_Xg=2fg$, then*

$$\mathcal{L}_X\lambda=(n-3)f\lambda+2Tr\big(d_A(i_XF_A)\wedge *F_A\big)$$

Proof. In local coordinates $\{x^i\}_{i=1}^n$, the n-form λ

$$\lambda = Tr(F_A \wedge *F_A) = \sum g^{ij} g^{kl} tr F_{ik} F_{jl} \sqrt{detg} dx^1 \wedge \ldots \wedge dx^n$$

is conformal of weight $n-3$, i.e. $\lambda(A, e^{2f}g) = e^{(n-3)f}\lambda(A,g)$ for any $f \in C^\infty$. The vector field X satisfies $\mathcal{L}_X g = 2fg$, so

$$F_t^* g = exp(2\int_0^t F_s^* f)g.$$

Since λ is conformal with wight $n-3$,

$$\begin{aligned}
(F_t^*\lambda)(A,g) &= \lambda(\tilde{F}_t^* A, F_t^* g) = exp\big((n-3)\int_0^t F_t^* f\big) \cdot \lambda(\tilde{F}_t^* A, g) \\
&= \big(1 + (n-3)tf + o(t^2)\big) \\
&\times Tr\big(F_A \wedge *F_A + 2t d_A(i_X F_A) \wedge *F_A + o(t^2)\big) \\
&= Tr(F_A \wedge *F_A + t(n-3)f F_A \wedge *F_A) \\
&+ 2tTr(d_A(i_X F_A) \wedge *F_A) + o(t^2)
\end{aligned}$$

where we used the fact $F_t^* f = f + o(t)$ and $\tilde{F}_t^* A = A + t i_X F_A + o(t^2)$. By the definition of Lie derivative

$$\mathcal{L}_X \lambda = \frac{d}{dt}(F_t^*\lambda)|_{t=0} = (n-3)f\lambda + 2Tr\big(d_A(i_X F_A) \wedge *F_A\big). \tag{4.2}$$

□

We consider $M = \mathbf{R} \times N$ with metric

$$g_M = e^{2t}(dt^2 + g_N)$$

where N is a compact Riemannian n-manifold, $n \geq 4$, with metric g_N. Then the vector field $X = \frac{\partial}{\partial t}$ satisfies

$$\mathcal{L}_X g_M = X \cdot e^{2t}(dt^2 + g_N) + e^{2t}(\mathcal{L}_X dt^2) = 2g_M,$$

and in this case, $f = 1$.

Theorem 4.2. *Let (M, g_M) be a Riemannian manifold as above. Let A be a Yang-Mills connection with L^2-bounded curvature F_A, i.e.*

$$\int_M |F_A|^2 < +\infty$$

over M. Then A must be a flat connection.

Proof. From (4.1) and (4.2), we have

$$\begin{aligned}\int_M \eta(n-3)\lambda &= -\int_M d\eta\wedge i_X\lambda - 2\int_M \eta Tr\big(d_A(i_XF_A)\wedge *F_A\big)\\ &= -\int_M d\eta\wedge i_X\lambda - 2\int_M Tr\big(d_A(\eta i_XF_A)\wedge *F_A\big)\\ &+ 2\int_M Tr\big((d\eta\wedge i_XF_A)\wedge *F_A\big)\\ &\leq 3\int_M |d\eta|\cdot|X|\cdot\lambda\end{aligned}\tag{4.3}$$

The second term in the second line vanishes since A is a Yang-Mills connection.

We choose the cut-off function with $\eta(t)=1$ on the interval $|t|\leq T$, $\eta(t)=0$ on the interval $|t|\geq 2T$, and $|d\eta|\leq 2T^{-1}$. Then $d\eta$ has support in $T\leq|t|\leq 2T$ and $|X(t)|=1$,

$$\int_M \eta(n-3)\lambda \leq \frac{6}{T}\int_{\{T\leq|t|\leq 2T\}\times N}\lambda.$$

Letting $T\to\infty$ we get

$$\int_M (n-3)\lambda = 0,$$

Then $F_A=0$. □

Acknowledgment

I would like to thank O. Lechtenfeld for kind comments regarding this and it companion article [1, 13] and S.K. Donaldson for helpful comments regarding his article [7], and H.N. Sá Earp for helpful comments regarding his article [20, 21]. This work is partially supported by Wu Wen-Tsun Key Laboratory of Mathematics of Chinese Academy of Sciences at USTC.

References

[1] Bauer, I., Ivanova, T.A., Lechtenfeld, O. and Lubbe, F., *Yang-Mills instantons and dyons on homogeneous G_2-manifolds.* JHEP. **2010**(10), 1–27 (2010)

[2] Blumenhagen, R., Körs, B., Lüst, D. and Stieberger, S., *Four-dimensional string compactifications with D-branes, orientifolds and fluxes.* Phys.Rept. **1** (445), 1–193 (2007)

[3] Bourguignon, J. P., Lawson, H.B., *Stability and isolation phenomena for Yang-Mills fields,* Comm. Math. Phys. **79**(2), 189–230 (1981)

[4] Bryant, R.L., *Some remarks on G_2-structures,* arXiv preprint math/0305124, (2003)

[5] Carrión, R.R., *A generalization of the notion of instanton,* Diff.Geom.Appl. **8**(1), 1–20 (1998)

[6] Corrigan, E. ,Devchand, C.,Fairlie, D.B., Nuyts. J., *First order equations for gauge fields in spaces of dimension great than four,* Nucl.Phys.B, **214**(3), 452–464 (1983)

[7] Donaldson, S.K., *Floer homology groups in Yang-Mills theory*, Cambridge University Press, (2002)

[8] Donaldson, S.K., Thomas R.P., *Gauge theory in higher dimensions*, The Geometric Universe, Oxford, 31–47 (1998)

[9] Donaldson, S.K., Segal. E., *Gauge theory in higher dimensions, II.* arXiv:0902.3239, (2009)

[10] Douglas, M.R., Kachru, S., *Flux compactification.* Rev.Mod.Phys. **79**(2):733 (2007)

[11] Graña, M.,*Flux compactifications in string theory: A comprehensive review.* Phys.Rept. **423**(3), 91–158 (2006)

[12] Green, M.B., Schwarz, J.H. and Witten, E., *Supperstring theory*, Cambridge University Press, (1987)

[13] Harland, D., Ivanova, T.A., Lechtenfeld, O. and Popov, A.D., *Yang-Mills flows on nearly Kähler manifolds and G_2-instantons.* Comm.Math.Phys. **300**(1), 185–204 (2010)

[14] Harland, D., Nölle .C, *Instantons and Killing spinors*, JHEP. **3**, 1–38 (2012)

[15] Ivanova. T.A., Popov, A.D., *Instantons on special holonomy manifolds*, Phys.Rev.D **85**(10) (2012)

[16] Ivanova, T.A., Lechtenfeld, O., Popov, A.D. and Rahn, T., *Instantons and Yang-Mills flows on coset spaces.* Lett. Math. Phys. **89** (3), 231–247 (2009)

[17] Li, P., Schoen, R., *L^p and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Mathematica, **153**(1), 279–301 (1984)

[18] Morrey, C.B., *Multiple integrals in the calculus of variations*, Springer,(1966)

[19] Parker, T., *Conformal fields and stability*, Math.Z. **185**(3) 305–319 (1984)

[20] Sá Earp, H.N., *Instantons on G_2-manifolds*, London Ph.D Thesis, (2009)

[21] Sá Earp, H.N., *Generalised Chern-Simons Theory and G_2-Instantons over Associative Fibrations.* SIGMA, **10**:083 (2014)

[22] Tian, G., *Gauge theory and calibrated geometry, I.* Ann.Math. **151**(1), 193–268 (2000)

[23] Ward, R.S., *Completely solvable gauge field equations in dimension great than four*, Nucl.Phys.B **236**(2), 381–396 (1984)